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The conjugacy classes of the involutive automorphisms of affine Kac–Moody algebra $A_2^{(1)}$

J F Cornwell

Department of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, UK

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Abstract. All the conjugacy classes of involutive automorphisms of the affine Kac–Moody algebra $A_2^{(1)}$ are obtained using the matrix formulation of automorphisms of an affine Kac–Moody algebra that was developed in a previous paper.

1. Introduction

A general matrix formulation of the automorphisms of untwisted affine Kac–Moody algebras was developed in a previous paper [1], and subsequently applied to the special case of $A_1^{(1)}$ [2]. (These papers will henceforth be referred to as papers I and II respectively.) However, as was noted in paper II, $A_1^{(1)}$ has some special features that are absent in $A_l^{(1)}$ with $l > 1$, and these special features help to simplify the analysis. As $A_2^{(1)}$ also has some special features which makes it different from $A_l^{(1)}$ with $l > 2$, and as the algebras $A_l^{(1)}$ with $l > 2$ are quite complicated, it is also worthwhile treating the case $A_2^{(1)}$ separately. This is the object of the present paper, which will be devoted to finding the conjugacy classes of the involutive automorphisms of $A_2^{(1)}$.

The notations and conventions that will be employed in the present paper are those defined in paper I (with the additional convention here that equation labels such as (7) and (I.7) refer to the seventh numbered equation of the present paper and of paper I respectively). When the untwisted affine Kac–Moody algebra $\tilde{\mathcal{L}}$ is $A_2^{(1)}$, the corresponding simple Lie algebra is A_2, for which the rank l has value 2. The generalized Cartan matrix of $A_2^{(1)}$ is

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} . \tag{1}$$

The three simple roots of $A_2^{(1)}$ are α_0, α_1 and α_2, and as the highest root α_H^0 of A_2 is $\alpha_1^0 + \alpha_2^0$, the root imaginary δ is given by $\delta = \alpha_0 + \alpha_1 + \alpha_2$, and so $c = h_\delta = h_{\alpha_0} + h_{\alpha_1} + h_{\alpha_2}$. For these simple roots

$$\langle\alpha_0, \alpha_0\rangle = \langle\alpha_1, \alpha_1\rangle = \langle\alpha_2, \alpha_2\rangle = \langle\alpha_1^0, \alpha_1^0\rangle^0 = \langle\alpha_2^0, \alpha_2^0\rangle^0 = \tfrac{1}{3} \tag{2}$$

and

$$\langle\alpha_0,\alpha_1\rangle = \langle\alpha_1,\alpha_2\rangle = \langle\alpha_2,\alpha_0\rangle = \langle\alpha_1^0,\alpha_2^0\rangle^0 = -\tfrac{1}{6} \tag{3}$$

with $\langle\alpha_k,\alpha_{k'}\rangle = \langle\alpha_{k'},\alpha_k\rangle$ for $k,k' = 0,1,2$ with $k \neq k'$.

Let $\mathbf{\Gamma}$ be the three-dimensional irreducible representation of A_2 in which

$$\mathbf{\Gamma}(h^0_{\alpha_1^0}) = \mathbf{h}^0_{\alpha_1^0} = \tfrac{1}{6}(\mathbf{e}_{11} - \mathbf{e}_{22}) \qquad \mathbf{\Gamma}(h^0_{\alpha_2^0}) = \mathbf{h}^0_{\alpha_2^0} = \tfrac{1}{6}(\mathbf{e}_{22} - \mathbf{e}_{33}) \tag{4}$$

$$\mathbf{\Gamma}(e^0_{\alpha_1^0}) = \mathbf{e}^0_{\alpha_1^0} = (\tfrac{1}{6})^{1/2}\mathbf{e}_{12} \qquad \mathbf{\Gamma}(e^0_{-\alpha_1^0}) = \mathbf{e}^0_{-\alpha_1^0} = -(\tfrac{1}{6})^{1/2}\mathbf{e}_{21} \tag{5}$$

$$\mathbf{\Gamma}(e^0_{\alpha_2^0}) = \mathbf{e}^0_{\alpha_2^0} = (\tfrac{1}{6})^{1/2}\mathbf{e}_{23} \qquad \mathbf{\Gamma}(e^0_{-\alpha_2^0}) = \mathbf{e}^0_{-\alpha_2^0} = -(\tfrac{1}{6})^{1/2}\mathbf{e}_{32} \tag{6}$$

$$\mathbf{\Gamma}(e^0_{\alpha_1^0+\alpha_2^0}) = \mathbf{e}^0_{\alpha_1^0+\alpha_2^0} = (\tfrac{1}{6})^{1/2}\mathbf{e}_{13} \tag{7}$$

$$\mathbf{\Gamma}(e^0_{-\alpha_1^0-\alpha_2^0}) = \mathbf{e}^0_{-\alpha_1^0-\alpha_2^0} = -(\tfrac{1}{6})^{1/2}\mathbf{e}_{31} \tag{8}$$

where the $\mathbf{e}_{jk}$ are the 3×3 matrices defined by $(\mathbf{e}_{jk})_{rs} = \delta_{jr}\delta_{ks}$ for $j,k,r,s = 1,2,3$. The value of the Dynkin index of this representation is given by $\gamma = \frac{1}{6}$.

By contrast to the corresponding situation for $A_1^{(1)}$, this representation of $A_2^{(1)}$ is *not* equivalent to its contragredient representation. Thus for $A_2^{(1)}$ the set of type 1b involutive automorphisms does not coincide with the set of type 1a involutive automorphisms and the set of type 2b involutive automorphisms does not coincide with the set of type 2a involutive automorphisms. The type 1a and 1b sets both divide into two disjoint subsets with $u = 1$ and $u = -1$, whereas for the determination of the representatives of the conjugacy classes of the type 2a and the type 2b sets it is sufficient to let $u = 1$. Consequently there are six sets of involutive automorphisms of $A_2^{(1)}$ to be considered. These are:

(i) type 1a involutive automorphisms with $u = 1$, which will be analysed in section 2;

(ii) type 1a involutive automorphisms with $u = -1$, which will be analysed in section 3;

(iii) type 1b involutive automorphisms with $u = 1$, which will be analysed in section 4;

(iv) type 1b involutive automorphisms with $u = -1$, which will be analysed in section 5;

(v) type 2a involutive automorphisms (with $u = 1$), which will be analysed in section 6;

(vi) type 2b involutive automorphisms (with $u = 1$), which will be analysed in section 7.

The conclusions concerning involutive automorphisms of $A_2^{(1)}$ are summarized in section 8, and compared there with previous related work.

One of the conclusions of the analysis of $A_1^{(1)}$ given in paper II was that one can rely completely on the matrix formulation of automorphisms, the only structural ingredient needed being some knowledge of the root transformations of the corresponding simple Lie algebra A_1. As mentioned in section 3 of paper I, every conjugacy class of involutive automorphisms of $\tilde{\mathcal{L}}$ contains at least one Cartan-preserving involutive automorphism, and each such Cartan-preserving involutive automorphism is associated with an involutive root transformation τ^0 of $\tilde{\mathcal{L}}$. Thus for the determination of the conjugacy classes of $A_2^{(1)}$ the relevant information that is needed is that the corresponding simple Lie algebra A_2 has four conjugacy classes of involutive root transformations τ^0, whose representatives may be taken to be:

(i) $\tau^0 = E$ (the identity), for which

$$\tau^0(\alpha_1^0) = \alpha_1^0 \qquad \tau^0(\alpha_2^0) = \alpha_2^0 \tag{9}$$

(ii) $\tau^0 = S^0_{\alpha_1^0}$, for which

$$\tau^0(\alpha_1^0) = -\alpha_1^0 \qquad \tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0 \tag{10}$$

(iii) $\tau^0 = \rho^0$ (the Dynkin diagram symmetry operation for A_2), for which

$$\tau^0(\alpha_1^0) = \alpha_2^0 \qquad \tau^0(\alpha_2^0) = \alpha_1^0 \tag{11}$$

(iv) $\tau^0 = \tau^0_{\mathrm{Cartan}}$ (the Cartan involution for A_2), for which

$$\tau^0(\alpha_1^0) = -\alpha_1^0 \qquad \tau^0(\alpha_2^0) = -\alpha_2^0. \tag{12}$$

In fact detailed investigation shows that the group of root-preserving transformations of A_2 consists of the following 12 elements:

(i) the identity;

(ii) the seven two-fold transformations, which fall into three conjugacy classes:

(a) the Weyl reflections $S^0_{\alpha_1^0}$, $S^0_{\alpha_2^0}$ and $S^0_{\alpha_1^0+\alpha_2^0}$;

(b) ρ^0, $S^0_{\alpha_1^0} \circ \tau^0_{\mathrm{Cartan}}$ and $S^0_{\alpha_2^0} \circ \tau^0_{\mathrm{Cartan}}$;

(c) τ^0_{Cartan};

(iii) the two three-fold transformations $S^0_{\alpha_1^0} \circ S^0_{\alpha_2^0}$ and $S^0_{\alpha_2^0} \circ S^0_{\alpha_1^0}$;

(iv) the two six-fold transformations $S^0_{\alpha_1^0} \circ \rho^0$ and $S^0_{\alpha_2^0} \circ \rho^0$.

As will be seen in a subsequent paper, the group of root-preserving transformations of A_l has a quite different structure for $l \geqslant 3$.

2. Study of the involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = 1$

2.1. Determination of the involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = 1$

It is easily shown that there exists no non-singular 3×3 matrix $\mathsf{U}(t)$ that satisfies both of the conditions:

$$\begin{aligned} \mathsf{U}(t)\mathsf{h}^0_{\alpha_1^0}\mathsf{U}(t)^{-1} &= \mathsf{h}^0_{\alpha_2^0} \\ \mathsf{U}(t)\mathsf{h}^0_{\alpha_2^0}\mathsf{U}(t)^{-1} &= \mathsf{h}^0_{\alpha_1^0}. \end{aligned} \tag{13}$$

Consequently there are *no* type 1a involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (11). Similarly there are *no* type 1a involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (12), so attention may be concentrated on those associated with the remaining two root transformations listed at the end of the previous section.

2.1.1. Involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$. The most general 3×3 matrix $\mathbf{U}(t)$ that satisfies

$$\begin{aligned}\mathbf{U}(t)\mathbf{h}^0_{\alpha_1^0}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_1^0}\\ \mathbf{U}(t)\mathbf{h}^0_{\alpha_2^0}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_2^0}\end{aligned} \tag{14}$$

with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by $\mathbf{U}(t) = \mathrm{diag}(\eta_1' t^{k_1'}, \eta_2' t^{k_2'}, \eta_3' t^{k_3'})$, where η_1', η_2' and η_3' are arbitrary non-zero complex numbers and k_1', k_2' and k_3' are arbitrary integers. However, (I.196) shows that $(\eta_1')^{-1}t^{-k_1'}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta_k')/(\eta_1')$ and $k_k = k_k' - k_1'$ for $k = 2, 3$, it follows that the most general automorphism of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$ corresponds to

$$\mathbf{U}(t) = \mathrm{diag}(1, \eta_2 t^{k_2}, \eta_3 t^{k_3}) \tag{15}$$

where η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.136) now reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$k_2 = k_3 = 0 \qquad \eta_2 = \pm 1 \qquad \text{and} \qquad \eta_3 = \pm 1. \tag{16}$$

Thus there are *only four* involutive automorphisms of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$.

2.1.2. Involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$. The most general 3×3 matrix $\mathbf{U}(t)$ that satisfies

$$\begin{aligned}\mathbf{U}(t)\mathbf{h}^0_{\alpha_1^0}\mathbf{U}(t)^{-1} &= -\mathbf{h}^0_{\alpha_1^0}\\ \mathbf{U}(t)\mathbf{h}^0_{\alpha_2^0}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_1^0} + \mathbf{h}^0_{\alpha_2^0}\end{aligned} \tag{17}$$

with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by $\mathbf{U}(t) = \eta_1' t^{k_1'}\mathbf{e}_{12} + \eta_2' t^{k_2'}\mathbf{e}_{21} + \eta_3' t^{k_3'}\mathbf{e}_{33}$, where η_1', η_2' and η_3' are arbitrary non-zero complex numbers and k_1', k_2' and k_3' are arbitrary integers. However, (I.196) shows that $(\eta_1')^{-1}t^{-k_1'}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta_k')/(\eta_1')$ and $k_k = k_k' - k_1'$ for $k = 2, 3$, it follows that the most general automorphism of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$ corresponds to

$$\mathbf{U}(t) = \mathbf{e}_{12} + \eta_2 t^{k_2}\mathbf{e}_{21} + \eta_3 t^{k_3}\mathbf{e}_{33} \tag{18}$$

where η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.136) again reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$k_2 = 2k_3 \qquad \text{and} \qquad \eta_2 = (\eta_3)^2. \tag{19}$$

2.2. Identification of conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = 1$

It will now be shown that there are *two* conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = 1$. These two classes contain the following Cartan-preserving automorphisms:

(i) the identity automorphism, which corresponds to

$$\mathbf{U}(t) = 1 \qquad u = 1 \qquad \xi = 0 \tag{20}$$

and which is in a class of its own;

(ii) all the remaining involutive automorphisms mentioned earlier in this section, for which a representative may be taken to be the type 1a involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(h_{\alpha_2}) = h_{\alpha_2} \qquad \psi(c) = c \\ &\psi(d) = d \qquad \psi(e_{\pm\alpha_0}) = -e_{\pm\alpha_0} \\ &\psi(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi(e_{\pm\alpha_2}) = -e_{\pm\alpha_2} \end{aligned} \tag{21}$$

which corresponds (by (I.67), (I.71), (I.73) and (I.138)) to

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, -1) \qquad u = 1 \qquad \xi = 0. \tag{22}$$

It is clear that the identity automorphism is in a class of its own, so attention will be concentrated on establishing that *all* the remaining involutive automorphisms mentioned earlier in this section are indeed mutually conjugate.

First, as

$$\mathrm{diag}(1, 1, -1) = (\mathbf{e}_{11} + \mathbf{e}_{23} + \mathbf{e}_{32})\,\mathrm{diag}(1, -1, 1)(\mathbf{e}_{11} + \mathbf{e}_{23} + \mathbf{e}_{32})^{-1} \tag{23}$$

and

$$\mathrm{diag}(-1, -1, 1) = (\mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31})\,\mathrm{diag}(1, -1, -1)(\mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31})^{-1} \tag{24}$$

it follows from the conjugacy condition (I.158) that the type 1a involutive automorphisms corresponding to $u = 1$, $\xi = 0$ and to the matrices $\mathbf{U}(t)$ of (15) with $\eta_2 = -1$, $\eta_3 = 1$, $k_2 = k_3 = 0$ and with $\eta_2 = -1$, $\eta_3 = -1$, $k_2 = k_3 = 0$ are both conjugate to the type 1a involutive automorphism (21) corresponding to (22) via type 1a automorphisms with $s = 1$.

Second, as (19) implies that

$$\eta_3 t^{k_3}\,\mathrm{diag}(1, 1, -1) = \mathbf{S}(t)(\mathbf{e}_{12} + \eta_2 t^{k_2}\mathbf{e}_{21} + \eta_3 t^{k_3}\mathbf{e}_{33})\mathbf{S}(t)^{-1}$$

with

$$\mathbf{S}(t) = \eta_3 t^{k_3}\mathbf{e}_{11} + \mathbf{e}_{12} + \sqrt{2}\mathbf{e}_{23} + \eta_3 t^{k_3}\mathbf{e}_{31} - \mathbf{e}_{32} \tag{25}$$

it follows from the conjugacy condition (I.158) that in every case the type 1a involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (18) with η_2, η_3, k_2 and k_3 satisfying (19) is conjugate to the type 1a involutive automorphism (21) corresponding to (22) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (25) and to $s = 1$.

3. Study of the involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = -1$

3.1. *Determination of the involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = -1$*

As there exists no non-singular 3×3 matrix $\mathbf{U}(t)$ that satisfies both of the conditions (13), it follows that there are *no* type 1a involutive automorphisms of $A_2^{(1)}$ with $u = -1$ corresponding to the root transformation (11). Similarly there are *no* type 1a involutive automorphisms of $A_2^{(1)}$ with $u = -1$ corresponding to the root transformation (12), so attention may be concentrated on those associated with the remaining two root transformations listed at the end of section 1.

3.1.1. Involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$. Consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (14) leads to the conclusion that the most general automorphism of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$ corresponds to (15), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.136) now reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$k_2 = k_3 = 0 \qquad \eta_2 = \pm 1 \qquad \text{and} \qquad \eta_3 = \pm 1. \tag{26}$$

Thus there are *only four* involutive automorphisms of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$.

3.1.2. Involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$. Consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (17) leads to the conclusion that the most general automorphism of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$ corresponds to (18), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.136) now reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$k_2 = 2k_3 \qquad \text{and} \qquad \eta_2 = (-1)^{k_3}(\eta_3)^2. \tag{27}$$

3.2. *Identification of conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = -1$*

It will now be shown that there is *only one* conjugacy class of involutive automorphisms of $A_2^{(1)}$ of type 1a with $u = -1$. This contains all the involutive automorphisms mentioned earlier in this section. A representative may be taken to be the type 1a involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(h_{\alpha_2}) = h_{\alpha_2} \qquad \psi(c) = c \\ &\psi(d) = d \qquad \psi(e_{\pm\alpha_0}) = e_{\pm\alpha_0} \\ &\psi(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi(e_{\pm\alpha_2}) = -e_{\pm\alpha_2} \end{aligned} \tag{28}$$

which corresponds (by (I.67), (I.71), (I.73) and (I.138)) to

$$\mathbf{U}(t) = \mathrm{diag}(1,1,-1) \qquad u = -1 \qquad \xi = 0. \tag{29}$$

The first stage in establishing this result is to note that (23) can be recast in the form

$$\mathrm{diag}(1,1,-1) = \mathbf{S}(t)\,\mathrm{diag}(1,-1,1)\mathbf{S}(-t)^{-1}$$

where

$$\mathbf{S}(t) = \mathbf{S}(-t) = \mathbf{e}_{11} + \mathbf{e}_{23} + \mathbf{e}_{32}. \tag{30}$$

Similarly (24) can be recast in the form

$$\mathrm{diag}(-1,-1,1) = \mathbf{S}(t)\,\mathrm{diag}(1,-1,-1)\mathbf{S}(-t)^{-1}$$

where

$$\mathbf{S}(t) = \mathbf{S}(-t) = \mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31}. \tag{31}$$

It then follows from the conjugacy condition (I.158) (with $u_2 = -1$) that the type 1a involutive automorphisms corresponding to $u = -1$, $\xi = 0$, and to the matrices $\mathbf{U}(t)$ of (15) with $\eta_2 = -1, \eta_3 = 1,\ k_2 = k_3 = 0$ and with $\eta_2 = -1,\ \eta_3 = -1, k_2 = k_3 = 0$ are both conjugate to the type 1a involutive automorphism (28) corresponding to (29) via type 1a automorphisms belonging to the matrices $\mathbf{S}(t)$ of (30) and (31) and to $s = 1$.

The second stage is to note that as $\mathrm{diag}(-1,-1,1) = \mathbf{S}(t)\mathbf{1}\ \mathbf{S}(-t)^{-1}$, with

$$\mathbf{S}(t) = t\mathbf{e}_{12} + t\mathbf{e}_{21} + \mathbf{e}_{33} \tag{32}$$

it follows from the conjugacy condition (I.158) (with $u_2 = -1$) that the type 1a involutive automorphism corresponding to $u = -1$, $\xi = 0$, and to the matrix $\mathbf{U}(t) = \mathbf{1}$ is conjugate to the type 1a involutive automorphism (28) corresponding to (29) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (32) and to $s = 1$.

The final stage is to note that (27) implies that

$$\eta_3 t^{k_3}\,\mathrm{diag}(1,1,-1) = \mathbf{S}(t)\{\mathbf{e}_{12} + (-1)^{k_3}(\eta_3)^2 t^{2k_3}\mathbf{e}_{21} + \eta_3 t^{k_3}\mathbf{e}_{33}\}\mathbf{S}(-t)^{-1}$$

with the matrix $\mathbf{S}(t)$ of (25). Consequently the conjugacy condition (I.158) (with $u_2 = -1$) shows that in every case the type 1a involutive automorphism corresponding to $u = -1$ and to the matrix $\mathbf{U}(t)$ of (18) with η_2, η_3, k_2 and k_3 satisfying (27) is conjugate to the type 1a involutive automorphism (28) corresponding to (29) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (25) and to $s = 1$.

4. Study of the involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = 1$

4.1. Determination of the involutive automorphisms of $A_2^{(1)}$ *of type 1b with* $u = 1$

It is easily shown that there exists no non-singular 3×3 matrix $\mathbf{U}(t)$ that satisfies both of the conditions:

$$\begin{aligned}\mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_1^0}\}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_2^0}\\ \mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_2^0}\}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_1^0}.\end{aligned} \tag{33}$$

Consequently there are *no* type 1b involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (9). Similarly there are *no* type 1b involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (10), so attention may be concentrated on those associated with the remaining two root transformations (11) and (12) that are listed at the end of the section 1. (It will be noted that this set is the complement of the corresponding set of the 1a case.)

4.1.1. Involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$. The most general 3×3 matrix $\mathbf{U}(t)$ that satisfies

$$\begin{aligned} \mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_1^0}\}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_2^0} \\ \mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_2^0}\}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_1^0} \end{aligned} \tag{34}$$

with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by $\mathbf{U}(t) = \eta_1' t^{k_1'} \mathbf{e}_{13} + \eta_2' t^{k_2'} \mathbf{e}_{22} + \eta_3' t^{k_3'} \mathbf{e}_{31}$, where η_1', η_2' and η_3' are arbitrary non-zero complex numbers and k_1', k_2' and k_3' are arbitrary integers. However, (I.196) shows that $(\eta_1')^{-1} t^{-k_1'} \mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta_k')/(\eta_1')$ and $k_k = k_k' - k_1'$ for $k = 2, 3$, it follows that the most general automorphism of type 1b with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$ corresponds to

$$\mathbf{U}(t) = \mathbf{e}_{13} + \eta_2 t^{k_2} \mathbf{e}_{22} + \eta_3 t^{k_3} \mathbf{e}_{31} \tag{35}$$

where η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.139) now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_3 = 0$ and $\eta_3 = 1$, but imposes no additional constraints on k_2 and η_2. Thus the most general involutive automorphism of type 1b with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$ corresponds to

$$\mathbf{U}(t) = \mathbf{e}_{13} + \eta_2 t^{k_2} \mathbf{e}_{22} + \mathbf{e}_{31} \tag{36}$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer.

4.1.2. Involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = -\alpha_2^0$. The most general 3×3 matrix $\mathbf{U}(t)$ that satisfies

$$\begin{aligned} \mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_1^0}\}\mathbf{U}(t)^{-1} &= -\mathbf{h}^0_{\alpha_1^0} \\ \mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_2^0}\}\mathbf{U}(t)^{-1} &= -\mathbf{h}^0_{\alpha_2^0} \end{aligned} \tag{37}$$

with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by $\mathbf{U}(t) = \operatorname{diag}(\eta_1' t^{k_1'}, \eta_2' t^{k_2'}, \eta_3' t^{k_3'})$, where η_1', η_2' and η_3' are arbitrary non-zero complex numbers and k_1', k_2' and k_3' are arbitrary integers. However, (I.196) shows that $(\eta_1')^{-1} t^{-k_1'} \mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta_k')/(\eta_1')$ and $k_k = k_k' - k_1'$ for $k = 2, 3$, it follows that the most general automorphism of type 1b with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = -\alpha_2^0$ corresponds to

$$\mathbf{U}(t) = \operatorname{diag}(1, \eta_2 t^{k_2}, \eta_3 t^{k_3}) \tag{38}$$

where η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.139) again reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which imposes *no* additional constraints on k_2, k_3, η_2 and η_3.

4.2. Identification of conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = 1$

It will now be shown that there is *only one* conjugacy class of involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = 1$. This contains all the involutive automorphisms mentioned earlier in this section. A representative may be taken to be the type 1b involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_2} \\ &\psi(h_{\alpha_2}) = h_{\alpha_1} \qquad \psi(c) = c \\ &\psi(d) = d \qquad \psi(e_{\pm\alpha_0}) = -e_{\pm\alpha_0} \\ &\psi(e_{\pm\alpha_1}) = -e_{\pm\alpha_2} \qquad \psi(e_{\pm\alpha_2}) = -e_{\pm\alpha_1} \end{aligned} \tag{39}$$

which corresponds (by (I.68), (I.71), (I.73) and (I.141)) to

$$\mathbf{U}(t) = \mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31} \qquad u = 1 \qquad \xi = 0. \tag{40}$$

As this conclusion is significantly different from that obtained previously by Kobayashi [3], the argument will be given in some detail. The first stage in establishing this result is to note that as

$$\eta_2 t^{k_2}(\mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31}) = \mathbf{S}(t)(\mathbf{e}_{13} + \eta_2 t^{k_2}\mathbf{e}_{22} + \mathbf{e}_{31})\tilde{\mathbf{S}}(t)$$

with

$$\mathbf{S}(t) = \mathrm{diag}(\eta_2 t^{k_2}, 1, 1) \tag{41}$$

it follows from the conjugacy condition (I.166) that in every case the type 1b involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (36) (with η_2 an arbitrary non-zero complex number and k_2 an arbitrary integer) is conjugate to the type 1b involutive automorphism (39) corresponding to (40) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (41) and to $s = 1$.

The second stage is more complicated. Its aim is to show that in every case the type 1b involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (38) (where η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers) is conjugate to the type 1b involutive automorphism (39) corresponding to (40). The argument involves the following sequence of steps:

(i) As

$$\mathrm{diag}(1, t^{k_2-2\kappa_2}, t^{k_3-2\kappa_3}) = \mathbf{S}(t)\,\mathrm{diag}(1, \eta_2 t^{k_2}, \eta_3 t^{k_3})\tilde{\mathbf{S}}(t)$$

with

$$\mathbf{S}(t) = \mathrm{diag}(1, (\eta_2)^{-1/2}t^{-\kappa_2}, (\eta_3)^{-1/2}t^{-\kappa_3}) \tag{42}$$

where κ_2 and κ_3 are arbitrary integers, it follows from the conjugacy condition (I.166) that in every case the type 1b involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (38) is conjugate to a type 1b involutive automorphism corresponding to one of the following sets via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (42) and to $s = 1$:

$$\mathbf{U}(t) = \mathbf{1} \qquad u = 1 \tag{43}$$

$$\mathbf{U}(t) = \mathrm{diag}(1, t, 1) \qquad u = 1 \tag{44}$$

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, t) \qquad u = 1 \tag{45}$$

$$\mathbf{U}(t) = \mathrm{diag}(1, t, t) \qquad u = 1. \tag{46}$$

(ii) As

$$t\mathbf{1} = \mathbf{S}(t)\,\mathrm{diag}(1, t, 1)\tilde{\mathbf{S}}(t)$$

with

$$\mathbf{S}(t) = \tfrac{1}{2}\begin{pmatrix} t+1 & 0 & -\mathrm{i}(t-1) \\ 0 & 2 & 0 \\ \mathrm{i}(t-1) & 0 & t+1 \end{pmatrix} \tag{47}$$

whose inverse has entries that are all Laurent polynomials in t, as

$$\mathbf{S}(t)^{-1} = \tfrac{1}{2}\begin{pmatrix} t^{-1}+1 & 0 & -\mathrm{i}(t^{-1}-1) \\ 0 & 2 & 0 \\ \mathrm{i}(t^{-1}-1) & 0 & t^{-1}+1 \end{pmatrix} \tag{48}$$

it follows from the conjugacy condition (I.166) that the type 1b involutive automorphism corresponding to (44) is conjugate to the type 1b involutive automorphism corresponding to (43) via the type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (47) and to $s = 1$.

(iii) As $\mathrm{diag}(1, t, 1) = \mathbf{S}(t)\,\mathrm{diag}(1, 1, t)\tilde{\mathbf{S}}(t)$, with

$$\mathbf{S}(t) = \mathbf{e}_{11} + \mathbf{e}_{23} + \mathbf{e}_{32} \tag{49}$$

it follows from the conjugacy condition (I.166) that the type 1b involutive automorphism corresponding to (45) is conjugate to the type 1b involutive automorphism corresponding to (44) via the type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (49) and to $s = 1$, and thus, by the previous step, is conjugate to the type 1b involutive automorphism corresponding to (43).

(iv) As $t\,\mathrm{diag}(1, t, 1) = \mathbf{S}(t)\,\mathrm{diag}(1, t, t)\tilde{\mathbf{S}}(t)$, with

$$\mathbf{S}(t) = \mathbf{e}_{12} + t\mathbf{e}_{12} + \mathbf{e}_{33} \tag{50}$$

it follows from the conjugacy condition (I.166) that the type 1b involutive automorphism corresponding to (46) is conjugate to the type 1b involutive automorphism corresponding to (44) via the type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (50) and to $s = 1$, and thus is conjugate to the type 1b involutive automorphism corresponding to (43).

(v) The last step in this sequence is to note that as $\mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31} = \mathbf{S}(t)\mathbf{1}\tilde{\mathbf{S}}(t)$, with

$$\mathbf{S}(t) = \tfrac{1}{\sqrt{2}}\{i\mathbf{e}_{11} + \mathbf{e}_{13} + \sqrt{2}\mathbf{e}_{22} - i\mathbf{e}_{31} + \mathbf{e}_{33}\} \tag{51}$$

it follows from the conjugacy condition (I.166) that the type 1b involutive automorphism corresponding to (43) is conjugate to the type 1b involutive automorphism corresponding to (40) via the type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (51).

5. Study of the involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = -1$

5.1. *Determination of the involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = -1$*

As there exists no non-singular 3×3 matrix $\mathbf{U}(t)$ that satisfies both of the conditions (33), there are *no* type 1b involutive automorphisms of $A_2^{(1)}$ with $u = -1$ corresponding to the root transformation (9). Similarly there are *no* type 1b involutive automorphisms of $A_2^{(1)}$ with $u = -1$ corresponding to the root transformation (10), so again attention may be concentrated on those associated with the remaining two root transformations (11) and (12) that are listed at the end of the section 1. (Again this set is the complement of the corresponding set of the 1a case.)

5.1.1. Involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$. Consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (34) leads to the conclusion that the most general automorphism of type 1b with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$ corresponds to (35), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. However, the involutive condition (I.139) now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(-t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_3 = 0$ and $\eta_3 = (-1)^{k_2}$, but imposes no additional constraints on k_2 and η_2. Thus the most general involutive automorphism of type 1b with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$ corresponds to

$$\mathbf{U}(t) = \mathbf{e}_{13} + \eta_2 t^{k_2} \mathbf{e}_{22} + (-1)^{k_2} \mathbf{e}_{31} \tag{52}$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer.

5.1.2. Involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = -1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = -\alpha_2^0$. In this case consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (37) leads to the conclusion that the most general automorphism of type 1b with $u = -1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = -\alpha_2^0$ corresponds to (38), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.139) again reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(-t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which this time imposes *no* additional constraints on η_2 and η_3, but requires that both k_2 and k_3 must be *even* integers.

5.2. *Identification of conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = -1$*

It will now be shown that there is *only one* conjugacy class of involutive automorphisms of $A_2^{(1)}$ of type 1b with $u = -1$. This contains all the involutive automorphisms mentioned earlier in this section. A representative may be taken to be the type 1b involutive automorphism

$$\begin{gathered}
\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_2} \\
\psi(h_{\alpha_2}) = h_{\alpha_1} \qquad \psi(c) = c \\
\psi(d) = d \qquad \psi(e_{\pm\alpha_0}) = e_{\pm\alpha_0} \\
\psi(e_{\pm\alpha_1}) = -e_{\pm\alpha_2} \qquad \psi(e_{\pm\alpha_2}) = -e_{\pm\alpha_1}
\end{gathered} \tag{53}$$

which corresponds (by (I.68), (I.71), (I.73) and (I.141)) to

$$\mathbf{U}(t) = \mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31} \qquad u = -1 \qquad \xi = 0. \tag{54}$$

The first stage in establishing this result is to note that as

$$\eta_2 t^{k_2}\{\mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31}\} = \mathbf{S}(t)\{\mathbf{e}_{13} + \eta_2 t^{k_2}\mathbf{e}_{22} + (-1)^{k_2}\mathbf{e}_{31}\}\tilde{\mathbf{S}}(-t)$$

with $\mathbf{S}(t)$ given by (41), it follows from the conjugacy condition (I.166) (with $u_2 = -1$) that in every case the type 1b involutive automorphism corresponding to $u = -1$ and to the matrix $\mathbf{U}(t)$ of (36) (with η_2 an arbitrary non-zero complex number and k_2 an arbitrary integer) is conjugate to the type 1b involutive automorphism (53) corresponding to (54) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (41) and to $s = 1$.

The second stage is much simpler than that of the corresponding $u = 1$ case. As $\mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31} = \mathbf{S}(t)\,\mathrm{diag}(1, \eta_2 t^{k_2}, \eta_3 t^{k_3})\tilde{\mathbf{S}}(-t)$, with

$$\mathbf{S}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 0 & (\eta_3)^{-1/2}(-1)^{k_3/4}t^{-k_3/2} \\ 0 & \sqrt{2}(\eta_2)^{-1/2}(-1)^{k_2/4}t^{-k_2/2} & 0 \\ -\mathrm{i} & 0 & (\eta_3)^{-1/2}(-1)^{k_3/4}t^{-k_3/2} \end{pmatrix} \tag{55}$$

where k_2 and k_3 are arbitrary *even* integers, it follows from the conjugacy condition (I.166) (with $u_2 = -1$) that in every case the type 1b involutive automorphism corresponding to $u = -1$ and to the matrix $\mathbf{U}(t)$ of (38) (with η_2 and η_3 being arbitrary non-zero complex numbers and k_2 and k_3 being arbitrary *even* integers) is conjugate to the type 1b involutive automorphism (53) corresponding to (54) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (55) and to $s = 1$.

6. Study of the involutive automorphisms of $A_2^{(1)}$ of type 2a

6.1. Determination of the involutive automorphisms of $A_2^{(1)}$ of type 2a with $u = 1$

As there exists no non-singular 3×3 matrix $\mathbf{U}(t)$ that satisfies both of the conditions (13), it follows that there are *no* type 2a involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (11). Similarly there are *no* type 2a involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (12), so attention may be concentrated on those associated with the remaining two root transformations listed at the end of section 1.

6.1.1. Involutive automorphisms of $A_2^{(1)}$ of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$. In this case consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (14) leads to the conclusion that the most general automorphism of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$ corresponds to (15), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.142) now reduces to $\mathbf{U}(t)\mathbf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_2 = \pm 1 \qquad \text{and} \qquad \eta_3 = \pm 1 \tag{56}$$

but imposes no additional constraints on k_2 and k_3.

6.1.2. Involutive automorphisms of $A_2^{(1)}$ of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$. Consideration of the most general 3×3 matrix $\mathsf{U}(t)$ that satisfies (17) leads to the conclusion that the most general automorphism of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$ corresponds to (18), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.142) again reduces to $\mathsf{U}(t)\mathsf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_2 = 0$ and $\eta_2 = (\eta_3)^2$. Thus the most general involutive automorphism of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$ corresponds to

$$\mathsf{U}(t) = \mathbf{e}_{12} + (\eta_3)^2 \mathbf{e}_{21} + \eta_3 t^{k_3} \mathbf{e}_{33} \tag{57}$$

where η_3 is an arbitrary non-zero complex number and k_3 is an arbitrary integer.

6.2. *Identification of conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 2a*

It will now be shown that there are *three* conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 2a. Their representatives may be taken to be:

(i) the type 2a involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2h_{\alpha_1} - 2h_{\alpha_2} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(h_{\alpha_2}) = h_{\alpha_2} \qquad \psi(c) = -c \\ &\psi(d) = -d \qquad \psi(e_{\pm\alpha_0}) = e_{\pm(-\alpha_0 - 2\alpha_1 - 2\alpha_2)} \\ &\psi(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi(e_{\pm\alpha_2}) = e_{\pm\alpha_2} \end{aligned} \tag{58}$$

which corresponds (by (I.69), (I.71), (I.73) and (I.143)) to

$$\mathsf{U}(t) = \mathbf{1} \qquad u = 1 \qquad \xi = 0 \tag{59}$$

(ii) the type 2a involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2h_{\alpha_1} - 2h_{\alpha_2} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(h_{\alpha_2}) = h_{\alpha_2} \qquad \psi(c) = -c \\ &\psi(d) = -d \qquad \psi(e_{\pm\alpha_0}) = e_{\pm(-\alpha_0 - 2\alpha_1 - 2\alpha_2)} \\ &\psi(e_{\pm\alpha_1}) = -e_{\pm\alpha_1} \qquad \psi(e_{\pm\alpha_2}) = -e_{\pm\alpha_2} \end{aligned} \tag{60}$$

which corresponds (by (I.69), (I.71), (I.73) and (I.143)) to

$$\mathsf{U}(t) = \mathrm{diag}(1, -1, 1) \qquad u = 1 \qquad \xi = 0 \tag{61}$$

(iii) the type 2a involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_1} - h_{\alpha_2} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(h_{\alpha_2}) = -h_{\alpha_0} - h_{\alpha_1} \qquad \psi(c) = -c \\ &\psi(d) = -2h_{\alpha_1} - 4h_{\alpha_2} + 2c - d \qquad \psi(e_{\pm\alpha_0}) = e_{\pm(-\alpha_1 - \alpha_2)} \\ &\psi(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi(e_{\pm\alpha_2}) = e_{\pm(-\alpha_0 - \alpha_1)} \end{aligned} \tag{62}$$

which corresponds (by (I.69), (I.71), (I.73) and (I.143)) to

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, t) \qquad u = 1 \qquad \xi = 2. \tag{63}$$

The first stage in establishing this result is to show that it is true for the most general involutive automorphism of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_2^0$, which, as noted previously, corresponds to (15), where η_2 and η_3 satisfy (56) but there are no additional constaints on the integers k_2 and k_3. This argument itself has several parts:

(i) As

$$\mathrm{diag}(1, \eta_2 t^{k_2 - 2\kappa_2}, \eta_3 t^{k_3 - 2\kappa_3}) = \mathbf{S}(t)\, \mathrm{diag}(1, \eta_2 t^{k_2}, \eta_3 t^{k_3}) \mathbf{S}(t^{-1})^{-1}$$

where

$$\mathbf{S}(t) = \mathrm{diag}(1, t^{-\kappa_2}, t^{-\kappa_3}) \tag{64}$$

and where κ_2 and κ_3 are arbitrary integers, it follows from the conjugacy condition (I.174) that in every case the type 2a involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (15) is conjugate to a type 2a involutive automorphism corresponding to one of the following sets via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (64) and to $s = 1$:

$$\mathbf{U}(t) = \mathrm{diag}(1, \eta_2, \eta_3) \qquad u = 1 \tag{65}$$

$$\mathbf{U}(t) = \mathrm{diag}(1, \eta_2, \eta_3 t) \qquad u = 1 \tag{66}$$

$$\mathbf{U}(t) = \mathrm{diag}(1, \eta_2 t, \eta_3) \qquad u = 1 \tag{67}$$

$$\mathbf{U}(t) = \mathrm{diag}(1, \eta_2 t, \eta_3 t) \qquad u = 1. \tag{68}$$

(Here η_2 and η_3 satisfy (56).)

(ii) For the set of matrices $\mathbf{U}(t)$ of (65) there are *two* non-conjugate involutive automorphisms corresponding to (59) and (61). This follows for two reasons.

(a) The conjugacy conditions (I.174), (I.176), (I.178) and (I.180) applied to the type 2a automorphism with $\mathbf{U}_2(t) = \mathbf{1}$, $u_2 = 1$, and $s^{-2}u_2 = 1$ all reduce to $\eta t^k \mathbf{U}_1(t) = \mathbf{S}(t)\mathbf{S}(t^{-1})^{-1}$, which with $t = 1$ implies that $\eta \mathbf{U}_1(1) = \mathbf{1}$, so the type 2a involutive automorphism corresponding to (59) cannot be conjugate to any other involutive automorphism corresponding to the set (65).

(b) As $\mathrm{diag}(1, -1, 1) = \mathbf{S}(t)\, \mathrm{diag}(1, 1, -1) \mathbf{S}(t^{-1})^{-1}$, with $\mathbf{S}(t)$ given by

$$\mathbf{S}(t) = \mathbf{e}_{11} + \mathbf{e}_{23} + \mathbf{e}_{32} \tag{69}$$

and as $-\mathrm{diag}(1, -1, -1) = \mathbf{S}(t)\, \mathrm{diag}(1, 1, -1) \mathbf{S}(t^{-1})^{-1}$, with $\mathbf{S}(t)$ given by $\mathbf{S}(t) = \mathbf{e}_{13} + \mathbf{e}_{22} + \mathbf{e}_{31}$, it follows from the conjugacy condition (I.174) (with $u_2 = 1$) that the remaining type 2a involutive automorphisms corresponding to (65) are all conjugate to the type 2a involutive automorphism (61).

(iii) For the set of matrices $\mathbf{U}(t)$ of (66) there are *two* non-conjugate involutive automorphisms corresponding to (61) and (63). This follows for the following reasons.

(a) With $\mathbf{S}(t) = \mathbf{1}$ and $s = -1$ the conjugacy condition (I.174) implies that the $u = 1$ type 2a automorphisms with $\mathbf{U}(t) = \mathrm{diag}(1, 1, t)$ and $\mathbf{U}(t) = \mathrm{diag}(1, 1, -t)$ are conjugate.

(b) With $\mathbf{S}(t) = \mathbf{1}$ and $s = -1$ the conjugacy condition (I.174) implies that the $u = 1$ type 2a automorphisms with $\mathbf{U}(t) = \mathrm{diag}(1,-1,t)$ and $\mathbf{U}(t) = \mathrm{diag}(1,-1,-t)$ are conjugate.

(c) With $\mathbf{S}(t) = \frac{1}{2}\{(t+1)\mathbf{e}_{11} + (t-1)\mathbf{e}_{12} + (t-1)\mathbf{e}_{21} + (t+1)\mathbf{e}_{22} + 2\mathbf{e}_{33}\}$ and $s = 1$ the conjugacy condition (I.174) implies that the $u = 1$ type 2a automorphisms with $\mathbf{U}(t) = \mathrm{diag}(1,-1,t)$ and $\mathbf{U}(t) = \mathrm{diag}(1,-1,1)$ are conjugate.

(d) In the sets (65) and (66) the $u = 1$ type 2a automorphism with

$$\mathbf{U}(t) = \mathrm{diag}(1,1,-t) \tag{70}$$

is conjugate *only* to the $u = 1$ type 2a automorphism with $\mathbf{U}(t)$ given by (63). This follows because with $s = 1$ the conjugacy conditions (I.174), (I.176), (I.178) and (I.180) applied to the type 2a automorphism with $\mathbf{U}_2(t)$ given by (63) and $u_2 = 1$ all reduce to

$$\eta t^k \mathbf{U}_1(t) = \mathbf{S}(t)\mathbf{U}_2(t)\mathbf{S}(t^{-1})^{-1}. \tag{71}$$

With $t = 1$ (71) implies that $\eta \mathbf{U}_1(1) = \mathbf{1}$, which restricts the only possible matrices $\mathbf{U}_1(t)$ to be $\mathbf{1}$ and that of (63), but $\mathbf{U}_1(t) = \mathbf{1}$ does not satisfy (71) with $t = -1$. Similarly with $s = -1$ the conjugacy conditions (I.174), (I.176), (I.178) and (I.180) applied to the type 2a automorphism with $\mathbf{U}_2(t)$ given by (63) and $u_2 = 1$ all reduce to

$$\eta t^k \mathbf{U}_1(t) = \mathbf{S}(t)\mathbf{U}_2(-t)\mathbf{S}(t^{-1})^{-1}. \tag{72}$$

With $t = -1$ (72) implies that $\eta(-1)^k \mathbf{U}_1(-1) = \mathbf{1}$, which restricts the only possible matrices $\mathbf{U}_1(t)$ to be $\mathbf{1}$ and that of (70), but $\mathbf{U}_1(t) = \mathbf{1}$ does not satisfy (72) with $t = 1$.

(iv) On applying the conjugacy condition (I.174) with the matrix $\mathbf{S}(t)$ given by (69) (and $s = 1$) it follows immediately that all the involutive automorphisms corresponding to the set of matrices $\mathbf{U}(t)$ of (67) are conjugate to involutive automorphisms corresponding to the set of matrices $\mathbf{U}(t)$ of (66).

(v) As $(\eta_3 t)\,\mathrm{diag}(1,(\eta_2/\eta_3),(\eta_3)^{-1}t) = \mathbf{S}(t)\,\mathrm{diag}(1,\eta_2 t,\eta_3 t)\mathbf{S}(t^{-1})^{-1}$, with $\mathbf{S}(t)$ given by

$$\mathbf{S}(t) = \mathbf{e}_{13} + \mathbf{e}_{22} + t\mathbf{e}_{31} \tag{73}$$

on applying the conjugacy condition (I.174) with the matrix $\mathbf{S}(t)$ given by (73) (and $s = 1$) it follows immediately that all the involutive automorphisms corresponding to the set of matrices $\mathbf{U}(t)$ of (68) are conjugate to involutive automorphisms corresponding to members of the set of matrices $\mathbf{U}(t)$ of (66).

Now consider the involutive automorphisms of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0 + \alpha_2^0$, which, as noted previously, correspond to (57), where η_3 is an arbitrary non-zero complex number and k_3 is an arbitrary integer. As

$$(\eta_3 t^{k_3})\,\mathrm{diag}(1,-1,-1) = \mathbf{S}(t)\{\mathbf{e}_{12} + (\eta_3)^2\mathbf{e}_{21} + \eta_3 t^{k_3}\mathbf{e}_{33}\}\mathbf{S}(t^{-1})^{-1}$$

where

$$\mathbf{S}(t) = \tfrac{1}{\sqrt{2}}\{\eta_3 t^{k_3}\mathbf{e}_{11} + \mathbf{e}_{12} + \eta_3 t^{k_3}\mathbf{e}_{21} - \mathbf{e}_{22} + \sqrt{2}\mathbf{e}_{33}\} \tag{74}$$

it follows from the conjugacy condition (I.174) that in every case the type 2a involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (57) is conjugate to the type 2a involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (61).

7. Study of the involutive automorphisms of $A_2^{(1)}$ of type 2b

7.1. Determination of the involutive automorphisms of $A_2^{(1)}$ of type 2b with $u = 1$

Just as in the previous considerations of the type 1b automorphisms, as there exists no non-singular 3×3 matrix $\mathbf{U}(t)$ that satisfies both of the conditions (33), there are *no* type 2b involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (9). Similarly there are *no* type 2b involutive automorphisms of $A_2^{(1)}$ with $u = 1$ corresponding to the root transformation (10), so attention may be concentrated on those associated with the remaining two root transformations (11) and (12) that are listed at the end of the section 1. (This set is the complement of the corresponding set of the 2a case (and of the 1a case).)

7.1.1. Involutive automorphisms of $A_2^{(1)}$ of type 2b with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$. Consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (34) leads to the conclusion that the most general automorphism of type 2b with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$ corresponds to (35), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. However, the involutive condition (I.144) (with $u = 1$) now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t^{-1})^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_3 = 2k_2$ and $\eta_3 = 1$, but imposes no additional constraints on k_2 and η_2. Thus the most general involutive automorphism of type 2b with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_2^0$ and $\tau^0(\alpha_2^0) = \alpha_1^0$ corresponds to

$$\mathbf{U}(t) = \mathbf{e}_{13} + \eta_2 t^{k_2} \mathbf{e}_{22} + t^{2k_2} \mathbf{e}_{31} \qquad (75)$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer.

7.1.2. Involutive automorphisms of $A_2^{(1)}$ of type 2b with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = -\alpha_2^0$. In this case consideration of the most general 3×3 matrix $\mathbf{U}(t)$ that satisfies (37) leads to the conclusion that the most general automorphism of type 2b with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ and $\tau^0(\alpha_2^0) = -\alpha_2^0$ corresponds to (38), where again η_2 and η_3 are arbitrary non-zero complex numbers and k_2 and k_3 are arbitrary integers. The involutive condition (I.144) (with $u = 1$) again reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t^{-1})^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which this time imposes *no* additional constraints on η_2 and η_3, but requires that both $k_2 = 0$ and $k_3 = 0$. Thus

$$\mathbf{U}(t) = \mathrm{diag}(1, \eta_2, \eta_3) \qquad (76)$$

with η_2 and η_3 being arbitrary non-zero complex numbers.

7.2. Identification of conjugacy classes of involutive automorphisms of $A_2^{(1)}$ of type 2b

It will now be shown that there is *only one* conjugacy class of involutive automorphisms of $A_2^{(1)}$ of type 2b with $u = 1$. This contains all the involutive automorphisms mentioned earlier in this section. Its representative may be taken to be the Cartan involution

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = -h_{\alpha_1} \\ &\psi(h_{\alpha_2}) = -h_{\alpha_2} \qquad \psi(c) = -c \\ &\psi(d) = -d \qquad \psi(e_{\pm\alpha_0}) = e_{\mp\alpha_0} \\ &\psi(e_{\pm\alpha_1}) = e_{\mp\alpha_1} \qquad \psi(e_{\pm\alpha_2}) = e_{\mp\alpha_2} \end{aligned} \tag{77}$$

which corresponds (by (I.70), (I.71), (I.73) and (I.145)) to

$$\mathbf{U}(t) = \mathbf{1} \qquad u = 1 \qquad \xi = 0. \tag{78}$$

The first stage in establishing this result is to note that as $\eta_2 t^{k_2}\mathbf{1} = \mathbf{S}(t)\{\mathbf{e}_{13} + \eta_2 t^{k_2}\mathbf{e}_{22} + t^{2k_2}\mathbf{e}_{31}\}\tilde{\mathbf{S}}(t^{-1})$, with

$$\mathbf{S}(t) = (1/\sqrt{2})\{-\mathrm{i}\eta_2 t^{k_2}\mathbf{e}_{11} + \mathrm{i}\mathbf{e}_{13} + \sqrt{2}\mathbf{e}_{22} + \eta_2 t^{k_2}\mathbf{e}_{31} + \mathbf{e}_{33}\} \tag{79}$$

it follows from the conjugacy condition (I.182) that in every case the type 2b involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (75) (with η_2 an arbitrary non-zero complex number and k_2 an arbitrary integer) is conjugate to the type 2b involutive automorphism (77) corresponding to (78) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (79) and to $s = 1$.

The second stage is very simple. As $\mathbf{1} = \mathbf{S}(t)\,\mathrm{diag}(1, \eta_2, \eta_3)\tilde{\mathbf{S}}(t^{-1})$, with

$$\mathbf{S}(t) = \mathrm{diag}(1, (\eta_2)^{-1/2}, (\eta_3)^{-1/2}) \tag{80}$$

it follows from the conjugacy condition (I.182) that in every case the type 2b involutive automorphism corresponding to $u = 1$ and to the matrix $\mathbf{U}(t)$ of (76) (with η_2 and η_3 being arbitrary non-zero complex numbers) is conjugate to the type 2b involutive automorphism (77) corresponding to (78) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (80) and to $s = 1$.

8. Conclusions regarding the matrix formulation of the involutive automorphisms of $A_2^{(1)}$

The analysis of the previous sections shows that $A_2^{(1)}$ has *nine* conjugacy classes of involutive automorphisms. These are:

(i) the two conjugacy classes of type 1a involutive automorphisms with $u = 1$ listed in section 2.2, for which the representatives may be taken to be

(a) the identity automorphism, which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u and ξ being given by (20);

(b) the involutive automorphism (21), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u and ξ being given by (22);

(ii) the one conjugacy class of type 1a involutive automorphisms with $u = -1$ described in section 3.2, for which the representative may be taken to be the involutive automorphism (28), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u and ξ being given by (29);

(iii) the one conjugacy class of type 1b involutive automorphisms with $u = 1$ described in section 4.2, for which the representative may be taken to be the involutive automorphism (39), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u and ξ being given by (40);

(iv) the one conjugacy class of type 1b involutive automorphisms with $u = -1$ described in section 5.2, for which the representative may be taken to be the involutive automorphism (53), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u and ξ being given by (54);

(v) the three conjugacy classes of type 2a involutive automorphisms listed in section 6.2, for which the representatives may be taken to be

(a) the involutive automorphism (58), which corresponds to the type 2a automorphism with $\mathbf{U}(t)$, u and ξ being given by (59),

(b) the involutive automorphism (60), which corresponds to the type 2a automorphism with $\mathbf{U}(t)$, u and ξ being given by (61),

(c) the involutive automorphism (62), which corresponds to the type 2a automorphism with $\mathbf{U}(t)$, u and ξ being given by (63);

(vi) the one conjugacy class of type 2b involutive automorphisms described in section 7.2, for which the representative may be taken to be the Cartan involution (77), which corresponds to the type 2b automorphism with $\mathbf{U}(t)$, u and ξ being given by (78).

It is very interesting to compare these results with those obtained earlier by Kobayashi [3] for the derived algebra of $A_2^{(1)}$ by another method. (Of course, as Kobayashi considered *only* the *derived* algebra of $A_2^{(1)}$, his analysis did not include any discussion of the action of automorphisms on the scaling element d). Kobayashi [3] actually lists *ten* conjugacy classes of involutive automorphisms of $A_2^{(1)}$, made up of the identity automorphism and *nine* conjugacy classes of automorphisms of order 2. In Kobayashi's classification the nine order 2 automorphism conjugacy class representatives are called (a), (b$'$), (b$''$), (b$'''$), (c), (d), (d$'$), (e) and (f). The extensions of these to the whole of the Kac–Moody algebra $A_2^{(1)}$ will be now related to the conjugacy classes of $A_2^{(1)}$ listed at the beginning of this section. In particular it will be shown that the two involutive automorphisms (d) and (d$'$) of Kobayashi do actually lie in the *same* conjugacy class. After translation into the notations and conventions used above the extension of Kobayashi's involutive automorphisms are as follows.

(a) The extension ψ_{a} of Kobayashi's involutive automorphism (a) is

$$\psi_{\mathrm{a}}(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi_{\mathrm{a}}(h_{\alpha_1}) = h_{\alpha_1}$$
$$\psi_{\mathrm{a}}(h_{\alpha_2}) = h_{\alpha_2} \qquad \psi_{\mathrm{a}}(c) = c$$
$$\psi_{\mathrm{a}}(d) = d \qquad \psi_{\mathrm{a}}(e_{\pm\alpha_0}) = -e_{\pm\alpha_0}$$
$$\psi_{\mathrm{a}}(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi_{\mathrm{a}}(e_{\pm\alpha_2}) = -e_{\pm\alpha_2}$$

which corresponds (by (I.67), (I.71), (I.73) and (I.138)) to the type 1a automorphism with

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, -1) \qquad u = 1 \qquad \xi = 0$$

which is conjugate to the type 1a involutive automorphism (21) with $\mathbf{U}(t)$, u and ξ being given by (22).

(b′) The extension $\psi_{b'}$ of Kobayashi's involutive automorphism (b') is

$$\psi_{b'}(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi_{b'}(h_{\alpha_1}) = -h_{\alpha_0} - h_{\alpha_2}$$
$$\psi_{b'}(h_{\alpha_2}) = -h_{\alpha_0} - h_{\alpha_1} \qquad \psi_{b'}(c) = -c$$
$$\psi_{b'}(d) = 6h_{\alpha_1} + 6h_{\alpha_2} - 6c - d \qquad \psi_{b'}(e_{\pm\alpha_0}) = e_{\pm\alpha_0}$$
$$\psi_{b'}(e_{\pm\alpha_1}) = -e_{\mp(\alpha_0+\alpha_2)} \qquad \psi_{b'}(e_{\pm\alpha_2}) = -e_{\mp(\alpha_0+\alpha_1)}$$

which corresponds (by (I.70), (I.71), (I.73) and (I.145)) to the type 2a automorphism with

$$\mathsf{U}(t) = \mathrm{diag}(1, -t, t^2) \qquad u = 1 \qquad \xi = -6$$

which is conjugate to the type 1a involutive automorphism (62) with $\mathsf{U}(t)$, u and ξ being given by (63).

(b″) The extension $\psi_{b''}$ of Kobayashi's involutive automorphism (b″) is

$$\psi_{b''}(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi_{b''}(h_{\alpha_1}) = h_{\alpha_1}$$
$$\psi_{b''}(h_{\alpha_2}) = -2h_{\alpha_0} - 2h_{\alpha_1} + h_{\alpha_2} \qquad \psi_{b''}(c) = -c$$
$$\psi_{b''}(d) = 4h_{\alpha_1} + 8h_{\alpha_2} - 8c - d \qquad \psi_{b''}(e_{\pm\alpha_0}) = e_{\pm\alpha_0}$$
$$\psi_{b''}(e_{\pm\alpha_1}) = -e_{\pm\alpha_1} \qquad \psi_{b''}(e_{\pm\alpha_2}) = -e_{\mp(2\alpha_0+2\alpha_1+\alpha_2)}$$

which corresponds (by (I.70), (I.71), (I.73) and (I.145)) to the type 2a automorphism with

$$\mathsf{U}(t) = \mathrm{diag}(1, -1, t^2) \qquad u = 1 \qquad \xi = -8$$

which is conjugate to the type 1a involutive automorphism (60) with $\mathsf{U}(t)$, u and ξ being given by (61).

(b‴) The extension $\psi_{b'''}$ of Kobayashi's involutive automorphism (b‴) is

$$\psi_{b'''}(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi_{b'''}(h_{\alpha_1}) = h_{\alpha_1}$$
$$\psi_{b'''}(h_{\alpha_2}) = -2h_{\alpha_0} - 2h_{\alpha_1} + h_{\alpha_2} \qquad \psi_{b'''}(c) = -c$$
$$\psi_{b'''}(d) = 4h_{\alpha_1} + 8h_{\alpha_2} - 8c - d \qquad \psi_{b'''}(e_{\pm\alpha_0}) = e_{\pm\alpha_0}$$
$$\psi_{b'''}(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi_{b'''}(e_{\pm\alpha_2}) = e_{\mp(2\alpha_0+2\alpha_1+\alpha_2)}$$

which corresponds (by (I.70), (I.71), (I.73)) and (I.145)) to the type 2a automorphism with

$$\mathsf{U}(t) = \mathrm{diag}(1, 1, t^2) \qquad u = 1 \qquad \xi = -8$$

which is conjugate to the type 1a involutive automorphism (58) with $\mathsf{U}(t)$, u and ξ being given by (59).

(c) The extension ψ_c of Kobayashi's involutive automorphism (c) is

$$\psi_c(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi_c(h_{\alpha_1}) = h_{\alpha_1}$$
$$\psi_c(h_{\alpha_2}) = h_{\alpha_2} \qquad \psi_c(c) = c$$
$$\psi_c(d) = d \qquad \psi_c(e_{\pm\alpha_0}) = e_{\pm\alpha_0}$$
$$\psi_c(e_{\pm\alpha_1}) = e_{\pm\alpha_1} \qquad \psi_c(e_{\pm\alpha_2}) = -e_{\pm\alpha_2}$$

which corresponds (by (I.67), (I.71), (I.73) and (I.138)) to the type 1a automorphism with

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, -1 \qquad u = -1 \qquad \xi = 0$$

which is conjugate to the type 1a involutive automorphism (28) with $\mathbf{U}(t)$, u and ξ being given by (29).

(d) The extension ψ_{d} of Kobayashi's involutive automorphism (d) is

$$\psi_{\mathrm{d}}(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi_{\mathrm{d}}(h_{\alpha_1}) = -h_{\alpha_1}$$
$$\psi_{\mathrm{d}}(h_{\alpha_2}) = 2h_{\alpha_0} + 2h_{\alpha_1} + h_{\alpha_2} \qquad \psi_{\mathrm{d}}(c) = c$$
$$\psi_{\mathrm{d}}(d) = 4h_{\alpha_1} + 8h_{\alpha_2} - 8c + d \qquad \psi_{\mathrm{d}}(e_{\pm\alpha_0}) = e_{\mp\alpha_0}$$
$$\psi_{\mathrm{d}}(e_{\pm\alpha_1}) = e_{\mp\alpha_1} \qquad \psi_{\mathrm{d}}(e_{\pm\alpha_2}) = e_{\pm(2\alpha_0+2\alpha_1+\alpha_2)}$$

which corresponds (by (I.68), (I.71), (I.73) and (I.141)) to the type 1b automorphism with

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, t^2) \qquad u = 1 \qquad \xi = -8$$

which is conjugate to the type 1b involutive automorphism (39) with $\mathbf{U}(t)$, u and ξ being given by (40).

(d′) The extension $\psi_{\mathrm{d}'}$ of Kobayashi's involutive automorphism (d′) is

$$\psi_{\mathrm{d}'}(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi_{\mathrm{d}'}(h_{\alpha_1}) = h_{\alpha_0} + h_{\alpha_2}$$
$$\psi_{\mathrm{d}'}(h_{\alpha_2}) = h_{\alpha_0} + h_{\alpha_1} \qquad \psi_{\mathrm{d}'}(c) = c$$
$$\psi_{\mathrm{d}'}(d) = 6h_{\alpha_1} + 6h_{\alpha_2} - 6c + d \qquad \psi_{\mathrm{d}'}(e_{\pm\alpha_0}) = e_{\mp\alpha_0}$$
$$\psi_{\mathrm{d}'}(e_{\pm\alpha_1}) = -e_{\pm(\alpha_0+\alpha_2)} \qquad \psi_{\mathrm{d}'}(e_{\pm\alpha_2}) = -e_{\pm(\alpha_0+\alpha_1)}$$

which corresponds (by (I.68), (I.71), (I.73) and (I.141)) to the type 1b automorphism with

$$\mathbf{U}(t) = \mathrm{diag}(1, -t, t^2) \qquad u = 1 \qquad \xi = -6$$

which is *also* conjugate to the type 1b involutive automorphism (39) with $\mathbf{U}(t)$, u and ξ being given by (40).

(e) The extension ψ_{e} of Kobayashi's involutive automorphism (e) is the Cartan involution (77), which corresponds to the type 2b automorphism with $\mathbf{U}(t)$, u and ξ being given by (78).

(f) The extension ψ_{f} of Kobayashi's involutive automorphism (f) is

$$\psi_{\mathrm{f}}(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi_{\mathrm{f}}(h_{\alpha_1}) = -h_{\alpha_1}$$
$$\psi_{\mathrm{f}}(h_{\alpha_2}) = 2h_{\alpha_0} + 2h_{\alpha_1} + h_{\alpha_2} \qquad \psi_{\mathrm{f}}(c) = c$$
$$\psi_{\mathrm{f}}(d) = 4h_{\alpha_1} + 8h_{\alpha_2} - 8c + d \qquad \psi_{\mathrm{f}}(e_{\pm\alpha_0}) = e_{\mp\alpha_0}$$
$$\psi_{\mathrm{f}}(e_{\pm\alpha_1}) = e_{\mp\alpha_1} \qquad \psi_{\mathrm{f}}(e_{\pm\alpha_2}) = -e_{\pm(2\alpha_0+2\alpha_1+\alpha_2)}$$

which corresponds (by (I.68), (I.71), (I.73) and (I.141)) to the type 1b automorphism with

$$\mathbf{U}(t) = \mathrm{diag}(1, 1, -t^2) \qquad u = -1 \qquad \xi = -8$$

which is conjugate to the type 1b involutive automorphism (53) with $\mathbf{U}(t)$, u and ξ being given by (54).

Although the conjugacy of Kobayashi's involutive automorphisms (d) and (d′) is implied by the series of steps given in the analysis of section 4, it is still worthwhile exhibiting more explicitly the automorphism ϕ that is responsible. As

$$t\,\mathrm{diag}(1,1,t^2) = \mathbf{S}(t)\,\mathrm{diag}(1,-t,t^2)\tilde{\mathbf{S}}(t)$$

with

$$\mathbf{S}(t) = \frac{1}{2}\begin{pmatrix} t+1 & 0 & -\mathrm{i}t^{-1}(t-1) \\ 0 & 2\mathrm{i} & 0 \\ \mathrm{i}t(t-1) & 0 & t+1 \end{pmatrix} \tag{81}$$

it follows from the conjugacy condition (I.166) that $\psi_{\mathrm{d}} = \phi \circ \psi_{\mathrm{d}'} \circ \phi^{-1}$, where ϕ is the type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (81) and to $s = 1$. (It is easily checked that all the entries of $\mathbf{S}(t)^{-1}$ are also Laurent polynomials in the complex variable t.)

The involutive automorphisms of $A_2^{(1)}$ have also been considered previously by Levstein [4], but unfortunately his tables give insufficient information to allow a detailed comparison to be made with these results.

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